Received: *August 13, 2000*, Accepted: *October 10, 2000*

M(atrix)-theory scattering in the noncommutative $(2,0)$ theory

Ori J. Ganor and Joanna L. Karczmarek

Department of Physics, Jadwin Hall, Princeton University
NJ 08544, USA
E-mail: origa@viper.princeton.edu, joannak@princeton.edu

Abstract: We study the Quantum-Mechanics on the hyper-Kähler manifold that is the blow-up of an A_1-singularity. This system is relevant for M(atrix)-theory as it was conjectured to describe scattering in the "noncommutative" deformation of a free 5+1D tensor multiplet in the sector with two units of longitudinal light-like momentum.

Keywords: Field Theories in Higher Dimensions, M(atrix) Theories, Non-Commutative Geometry.

JHEP10(2000)024

Contents

1. Introduction

A single M5-brane is described at low-energies by a free tensor multiplet with $\mathcal{N} = (2, 0)$ supersymmetry. An extension of the free tensor multiplet to an interacting theory has been suggested in [1] and [2, 3]. It was motivated by the string-theory realization [4, 5] of gauge theories on a noncommutative $\mathbb{R}^4$. This conjectured 5+1D theory breaks Lorenz invariance explicitly. It is assumed to depend on a constant anti-self-dual 3-form parameter Θ^{ijk} of dimension Mass^{-3}. At low-energies the theory reduces to the free tensor multiplet.

In [6] a consistent limit of the M5-brane with a strong 3-form field-strength was presented and it was suggested that at low-energies this limit describes a decoupled "noncommutative" $(2, 0)$-theory.

Recently, the theory has been re-incarnated in OM-theory [7]. There, a limit with a critical time-like component, H_{012}, was studied and was argued to yield a decoupled theory. At low-energies it reduces to a free tensor multiplet. Presumably, this is the "noncommutative" $(2, 0)$ theory with nonzero $\Theta^{345} = \Theta^{012}$. In [7], time-like noncommutativity, critical electric fields and space-like noncommutativity are related in a unified structure. (See also [8]-[21] for related results.) The theory of [1, 2, 3] is a decoupled sector of a special case of OM-theory, for which Θ^{ijk} is light-like (see [21] for a nice discussion).

The action of a U(1) gauge-theory on a noncommutative $\mathbb{R}^4$, can be expanded as the free action plus terms that are of higher order (in the noncommutativity) and depend only on the field-strength F_{ij} and its derivatives [6]. Similarly, the equations of motion of the noncommutative 5+1D theory can, presumably, be expanded in Θ^{ijk}. We will derive the leading terms in section 2.

One of the exciting features about this interacting theory is that it has a conjectured M(atrix)-model [22] that is described in terms of a quantum mechanics on a nonsingular space [1, 2, 3].

The simplest nontrivial sector of this M(atrix)-model is the sector with longitudinal, light-cone, momentum of $p_{||} = 2/R_{||}$, where $R_{||}$ is the radius of the light-like direction. It describes the scattering of two massless particles, corresponding to the tensor multiplet, each with longitudinal, light-cone, momentum of $p_{||} = 1/R_{||}$. This is similar to the calculations of scattering of gravitons and their supersymmetric partners [23, 24, 25] in the M(atrix)-model of 10+1D M-theory. The M(atrix)-model for the noncommutative tensor multiplet with $p_{||} = 2/R_{||}$ is described by quantum-mechanics on a blown-up A_1-singularity. In this paper we will study this quantum-mechanics and calculate the low-energy scattering, in the quantum-mechanics.

The paper is organized as follows. In section 2 we describe the leading order low-energy limit of the noncommutative M5-brane theory. That is our motivation for studying the quantum-mechanics on a blown-up A_1 singularity. In section 3 we describe in details the Quantum Mechanics on the blown-up A_1 singularity. In subsection 3.3 we calculate the s-wave scattering amplitude on the A_1 singularity. In the appendix, we present the calculation for scattering of two scalar particles in field-theory, up to order $O(\Theta)^2$.

2. Motivation: noncommutative M5-brane

One motivation for studying the QM on the blown-up A_1-singularity is that it is the M(atrix)-model for the noncommutative deformation of a free 5+1D tensor-multiplet. We will now describe this theory at lowest order in the "noncommutativity."

2.1 Free M5-brane

A single M5-brane is described, at low-energies, by a free tensor multiplet of $\mathcal{N} = (2,0)$ supersymmetry. The bosonic fields are 5 free scalars, ϕ^I $(I = 1,\ldots,5)$ and a 3-form tensor field-strength H_{ijk} with equations of motion:

$$H_{ijk} = -\frac{1}{6}\epsilon_{ijklmn}H^{lmn}\,, \qquad \partial_{[l}H_{ijk]} = 0\,.$$

Here, indices are lowered and raised with the metric:

$$\eta_{mn} dx^m dx^n = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 ,$$

and the anti-symmetric ϵ-symbol is normalized such that: $\epsilon_{012345} = +1$. In particular, we have: $H_{012} = H_{345}$. The notation $[ij, \dots, k]$ means complete anti-symmetrization. Thus:

$$T_{[i_1 \dots i_n]} \equiv \frac{1}{n!} \sum_{\sigma \in S_n} (-)^\sigma T_{i_{\sigma(1)} \dots i_{\sigma(n)}} .$$

The normalization is such that on 3-cycles:

$$\int H_{ijk} dx_i dx_j dx_k \in 2\pi \mathbb{Z} .$$

Later on, we will need the propagator:

$$\langle H_{ijk}(x) H_{lmn}(y) \rangle = \int \frac{d^6 p}{(2\pi)^6} e^{ip(x-y)} G_{ijk;lmn}(p) . \tag{2.1}$$

We calculate it by adding a self-dual part to H_{ijk} and writing it as $H_{ijk} = 3\partial_{[i} B_{jk]}$ with the action (see for instance [26] for details):

$$-\frac{1}{24\pi} \int H_{ijk} H^{ijk} d^6 x .$$

We then keep only the anti-self-dual part of the propagator. The result is:

$$\begin{aligned} G_{ijk;lmn}(p) = {} & \frac{18\pi}{p^2 + i\epsilon} \eta_{rl} \eta_{sm} \eta_{tn} \left(p_{[i} p^{[r} \delta_j^s \delta_{k]}^{t]} - \frac{1}{6} \epsilon_{ijk}{}^{\sigma[st} p^{r]} \right) + \\ & + \frac{\pi}{2} \left(\epsilon_{ijklmn} - 6 \delta_{[i}^{[r} \delta_j^s \delta_{k]}^{t]} \eta_{lr} \eta_{ms} \eta_{nt} \right) . \end{aligned} \tag{2.2}$$

2.2 The interacting theory

The interacting theory is described by interactions $L_{int}(\Theta)$ that depend on a constant anti-self-dual 3-form Θ^{ijk}. It satisfies:

$$\Theta_{ijk} = -\frac{1}{6} \epsilon_{ijklmn} \Theta^{lmn} .$$

L_{int} involves the fields ϕ^I, H_{ijk}, the fermions and their derivatives.

To first order in Θ, the interactions can be described by a self-dual dimension-9 operator, $\mathcal{O}_{ijk}$, in the free theory. The interaction is $\delta L = \Theta^{ijk} \mathcal{O}_{ijk}$.

The bosonic part of the interaction turns out to be:

$$\delta L = \frac{1}{96\pi} \Theta^{ijk} H_{ijl} H^{mnl} H_{mnk} + \frac{1}{4} \Theta^{ikl} H_{jkl} \partial_i \Phi^I \partial^j \Phi^I + O(\Theta)^2 . \tag{2.3}$$

Here, H should not be confused with the critical asymptotic value of the tensor field-strength on the M5-brane in the construction of the theory from M-theory. The field H is fluctuating and is assumed to go to zero at infinity.

At order $O(\Theta)^2$, the scalar fields have a quartic interaction:

$$-\frac{\pi}{2} \eta_{kn} \Theta^{ijk} \Theta^{lmn} \partial_i \Phi^I \partial_j \Phi^J \partial_l \Phi^I \partial_m \Phi^J . \tag{2.4}$$

These terms can be determined by dimensional reduction, as we will now explain.

2.3 Dimensional reduction to 4+1D

If we compactify on S^1 of circumference $2\pi R$, the dimensional reduction to 4+1D proceeds according to:

$$F_{\mu\nu} = 2\pi R H_{\mu\nu 5}\,, \qquad \theta^{\mu\nu} = \Theta^{\mu\nu 5}\,, \qquad g^2 = 4\pi^2 R\,, \qquad \phi^I = (2\pi)^{3/2} R\Phi^I\,.$$

To first order in θ, the 4+1D action is [27, 13]:

$$\begin{aligned} L_{4+1D} = {} & \frac{1}{2g^2}\partial_\mu\phi^I\partial^\mu\phi^I + \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2g^2}\theta^{\mu\nu}F_{\nu\sigma}F^{\sigma\tau}F_{\tau\mu} - \\ & - \frac{1}{2g^2}\theta^{\mu\nu}F_{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} + \frac{1}{g^2}\theta^{\mu\nu}F_{\mu\sigma}\partial_\nu\phi^I\partial^\sigma\phi^I - \frac{1}{4g^2}\theta^{\mu\nu}F_{\mu\nu}\partial_\sigma\phi^I\partial^\sigma\phi^I\,. \end{aligned}$$

Assuming that supersymmetry protects the leading interactions from loop-corrections, we can obtain (2.3)–(2.4) by requiring that dimensional reduction should produce the latter corrections in 4+1D SYM.

3. Quantum-Mechanics on a blown-up A_1 singularity

In this section we will describe in detail the quantum mechanics on the target space — the blown up A_1-singularity. The quantum mechanics has $\mathcal{N} = 8$ supersymmetry and this is related to the hyper-Kähler structure on the target space. For a generic description of QM on hyper-Kähler manifolds, see [28].

3.1 The geometry

The M(atrix)-model of k coincident $M5$-branes in the sector with longitudinal momentum $p_{||} = N$ is postulated [1, 2, 3] to be described by Quantum Mechanics on the manifold $\mathcal{M}_{N,k}$ defined by

$$\left(\begin{array}{c} [X, X^\dagger] + [Y, Y^\dagger] + Z^\dagger Z - W^\dagger W = \xi^2 Id_{N\times N} \\ [X, Y] + Z^\dagger W = 0 \end{array}\right) \Big/ \,\mathrm{U}(N)\,, \tag{3.1}$$

where the group $\mathrm{U}(N)$ acts on the $N\times N$ complex matrices X and Y and on the $k\times N$ complex matrices Z and W in the natural way. For any N and k, the trace of matrices X and Y is a flat four-dimentional space $\mathbb{R}^4$ and so we can write $\mathcal{M}_{N,k} = \mathbb{R}^4 \times \tilde{\mathcal{M}}_{N,k}$. This flat four-dimensional part corresponds to the center-of-mass coordinates. We study a single M5-brane, $k = 1$, and $N = 2$. The manifold $\tilde{\mathcal{M}}_{N=2,k=1} \equiv \mathcal{M}$ is the blown-up A_1 singularity. It must be the same as $\tilde{\mathcal{M}}_{N=1,k=2}$, as can be confirmed explicitly. The metric is easier to obtain in the second case, for $k = 2, N = 1$, when $\mathcal{M}$ can be embedded in $\mathbb{C}^4$ as

$$\mathrm{Tr}\left(A^\dagger A\sigma_i\right) = \xi^2\delta_i^3\,, \qquad A \sim e^{i\epsilon}A\,, \tag{3.2}$$

where A is a 2×2 complex matrix and σ_i are the Pauli matrices.

These conditions (ignoring the U(1) quotient for now) can be satisfied by parametrizing A as follows

$$A = \xi e^{i\epsilon} g \begin{pmatrix} \cosh r & 0 \\ 0 & \sinh r \end{pmatrix}, \tag{3.3}$$

where g is an arbitrary element of SU(2), and ξ, r and ϵ are real. The induced metric on this five-(real)dimensional manifold is given by

$$\begin{aligned}\frac{\mathrm{Tr}\left(dA^\dagger dA\right)}{\xi^2} &= \cosh(2r)\left[dr^2 + \frac{1}{2}\,\mathrm{Tr}(dg^\dagger dg)\right] + \frac{1}{\cosh(2r)}\left[\begin{pmatrix}1 & 0\end{pmatrix} dg^\dagger g \begin{pmatrix}1\\0\end{pmatrix}\right]^2 + \\ &+ \cosh(2r)\left[d\epsilon + \frac{i}{\cosh(2r)}\begin{pmatrix}1 & 0\end{pmatrix} dg^\dagger g \begin{pmatrix}1\\0\end{pmatrix}\right]^2 \end{aligned} \tag{3.4}$$

To obtain the U(1) quotient, we must choose a function $\epsilon(r,g)$ such that the distance between $(r, g, \epsilon(r,g))$ and $(r+dr, g+dg, \epsilon(r+dr, g+dg))$ is minimized. This corresponds to choosing $\epsilon(r,g)$ in such a way that the last term in the above equation vanishes. Thus, the metric on $\mathcal{M}$ is simply

$$\frac{ds^2|_{\mathcal{M}}}{\xi^2} = \cosh(2r)\left[dr^2 + \frac{1}{2}\,\mathrm{Tr}(dg^\dagger dg)\right] + \frac{1}{\cosh(2r)}\left[\begin{pmatrix}1 & 0\end{pmatrix} dg^\dagger g \begin{pmatrix}1\\0\end{pmatrix}\right]^2. \tag{3.5}$$

It is apparent from equation (3.2) that $\mathcal{M}$ is invariant under SU(2) acting on A on the left and under U(1) acting on the right. g can be parametrized by three angles θ, ϕ and α in such a way as to make these symmetries explicit

$$g = \begin{pmatrix} \cos(\theta/2)e^{i(\alpha-\phi)/2} & -\sin(\theta/2)e^{i(-\alpha-\phi)/2} \\ \sin(\theta/2)e^{i(\alpha+\phi)/2} & \cos(\theta/2)e^{i(-\alpha+\phi)/2} \end{pmatrix} \tag{3.6}$$

θ runs from 0 to π, and ϕ and α run from 0 to 2π. Under this parametrization we find that the metric becomes

$$ds^2 = \xi^2\left\{\cosh(2r)dr^2 + \frac{\cosh(2r)}{4}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{\cosh^2(2r)-1}{4\cosh(2r)}(d\alpha - \cos\theta d\phi)^2\right\}. \tag{3.7}$$

It will be convenient to make the change of variables $\cosh(2r) = R^2$ (R runs from 1 to ∞)

$$ds^2 = \xi^2\left\{\frac{R^4}{R^4-1}dR^2 + \frac{R^2}{4}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{R^4-1}{4R^2}(d\alpha - \cos\theta d\phi)^2\right\} \tag{3.8}$$

It is apparent that, for large R, $\mathcal{M}$ approaches flat space, namely $\mathbb{R}^4/\mathbb{Z}_2$. This is the metric for the blown-up A_1 singularity.

For later reference, let us here record that the scalar laplacian is

$$\begin{aligned}\nabla^2 = \xi^{-2}\Bigg\{&\frac{1}{R^3}\partial_R\left(\frac{R^4-1}{R}\partial_R\right) + \\ &+ \frac{4}{R^2}\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) + \frac{1}{\sin^2\theta}\left(\partial_\phi^2 + \partial_\alpha^2 + 2\cos\theta\partial_\phi\partial_\alpha\right)\right] + \frac{4}{R^2(R^4-1)}\partial_\alpha^2\Bigg\}\end{aligned} \tag{3.9}$$

and that it is fully separable.

It is known that the manifold $\mathcal{M}$ is a hyper-Kähler manifold, a fact crucial to the existence of $\mathcal{N}=8$ supersymmetric QM. An explicit hyper-Kähler structure for $\mathcal{M}$ will now be given.

The following coordinates

$$a \equiv (R^4-1)^{1/4}\sin\frac{\theta}{2}\exp\left(\frac{i}{2}(\alpha+\phi)\right) \qquad b \equiv (R^4-1)^{1/4}\cos\frac{\theta}{2}\exp\left(\frac{i}{2}(\alpha-\phi)\right) \tag{3.10}$$

are good complex coordinates for $\mathcal{M}$. In these coordinates, the metric is

$$\begin{aligned}
g_{a\bar{a}} &= g_{\bar{a}a} = \frac{(a\bar{a}+b\bar{b})^3+b\bar{b}}{(a\bar{a}+b\bar{b})^2\sqrt{(a\bar{a}+b\bar{b})^2+1}}\\
g_{b\bar{b}} &= g_{\bar{b}b} = \frac{(a\bar{a}+b\bar{b})^3+a\bar{a}}{(a\bar{a}+b\bar{b})^2\sqrt{(a\bar{a}+b\bar{b})^2+1}}\\
g_{a\bar{b}} &= g_{\bar{b}a} = -\frac{\bar{a}b}{(a\bar{a}+b\bar{b})^2\sqrt{(a\bar{a}+b\bar{b})^2+1}}\\
g_{b\bar{a}} &= g_{\bar{a}b} = -\frac{a\bar{b}}{(a\bar{a}+b\bar{b})^2\sqrt{(a\bar{a}+b\bar{b})^2+1}}\,,
\end{aligned} \tag{3.11}$$

which is clearly hermitean and easily confirmed to be Kähler, with a Kähler potential

$$K = X - \tan^{-1}X \qquad \text{where} \qquad X = \sqrt{(a\bar{a}+b\bar{b})^2+1}\,. \tag{3.12}$$

Also, the determinant is $g=1$, so $\mathcal{M}$ is Ricci-flat, since $R_{i\bar{j}} = -\partial_i\partial_j \ln\det(g)$. The Kähler form, Ω_1, is as usual $(\Omega_1)_{i\bar{j}} = -(\Omega_1)_{\bar{j}i} = ig_{i\bar{j}}$. We have 2 more Kähler forms Ω_2 and Ω_3, satisfying the hyper-Kähler condition $g^{\alpha\gamma}(\Omega_a)_{\alpha\beta}(\Omega_b)_{\gamma\delta} = \epsilon_{abc}(\Omega_c)_{\beta\delta} + \delta_{ab}g_{\beta\delta}$, where $\alpha = i,\bar{j}$. Explicitly, these are (written in the complex coordinates $(a,b,\bar{a},\bar{b})$)

$$(\Omega_2)_{\alpha\beta} = \begin{pmatrix} 0 & i & 0 & 0\\ -i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{pmatrix} \qquad (\Omega_3)_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix} \tag{3.13}$$

Later, we will need a complex structure defined as usual by

$$(\Omega_2)_{\alpha\beta} = -g_{\alpha\gamma}J^{\gamma}_{\beta}\,. \tag{3.14}$$

3.2 Quantization

The construction of the $\mathcal{N}=8$ supersymmetric algebra and its representation in a coordinate basis presented in this section is general to any 2-(complex)dimensional hyper-Kähler manifold. We will first construct the $\mathcal{N}=4$ supersymmetric quantum algebra, following a procedure similar to that in [29], and then extent the supersymmetry to $\mathcal{N}=8$.

The lagrangian is obtained by dimensionally reducing the d=4, $\mathcal{N}=1$ chiral supergravity lagrangian in flat space [30]. Since after dimensional reduction the original SO$(3,1)$ spinor indices loose their meaning, we will use SU(2) spinors, making no distinction between dotted and undotted spinor indices. Writing SU(2) spinors on the right and SO$(3,1)$ spinors on the left (using the spinor conventions in [30]) we define

$$\chi^\alpha \equiv \chi^\alpha \qquad\qquad \bar{\chi}^\alpha \equiv -\bar{\chi}_{\dot{\alpha}} = \bar{\sigma}^{0\dot{\alpha}\alpha}\bar{\chi}_{\dot{\alpha}}\,. \tag{3.15}$$

The spinor products can be defined in our conventions as

$$\psi\chi \equiv \epsilon_{\alpha\beta}\psi^\alpha\chi^\beta = \psi^\alpha\chi_\alpha \qquad \bar{\psi}\chi \equiv \epsilon_{\alpha\beta}\bar{\psi}^\alpha\chi^\beta = \bar{\psi}^\alpha\chi_\alpha \qquad \bar{\psi}\bar{\chi} \equiv \epsilon_{\alpha\beta}\bar{\psi}^\alpha\bar{\chi}^\beta = \bar{\psi}^\alpha\bar{\chi}_\alpha\,. \tag{3.16}$$

Notice that this implies that $(\chi^\alpha)^\dagger = \bar{\chi}_\alpha$ and $(\chi_\alpha)^\dagger = -\bar{\chi}^\alpha$ which in turn gives

$$(\chi\phi)^\dagger = -\bar{\chi}\bar{\phi} \qquad (\bar{\chi}\bar{\phi})^\dagger = -\chi\phi \qquad (\bar{\chi}\phi)^\dagger = \bar{\chi}\phi\,. \tag{3.17}$$

Finally, following [30], $\epsilon^{12} = -\epsilon_{12} = 1$.

In this notation, the dimensionally reduced lagrangian is

$$\begin{aligned}
\mathcal{L} &= g_{i\bar{j}}\dot{z}^i\dot{\bar{z}}^{\bar{j}} - ig_{i\bar{j}}\bar{\chi}^{\bar{j}}(D_t\chi^i) - \frac{1}{4}R_{i\bar{j}k\bar{l}}(\chi^i\chi^k)\left(\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}}\right)\\
&\sim g_{i\bar{j}}\dot{z}^i\dot{\bar{z}}^{\bar{j}} - \frac{i}{2}g_{i\bar{j}}\bar{\chi}^{\bar{j}}(D_t\chi^i) + \frac{i}{2}g_{i\bar{j}}\left(D_t\bar{\chi}^{\bar{j}}\right)\chi^i - \frac{1}{4}R_{i\bar{j}k\bar{l}}(\chi^i\chi^k)\left(\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}}\right)
\end{aligned} \tag{3.18}$$

where $D_t\chi^i = \dot{\chi} + \Gamma^i_{ab}\chi^a\dot{z}^b$ and the two forms of the lagrangian differ only by a total time derivative. We will use the second form, which is real.

The conjugate momenta are

$$\begin{aligned}
P_i &\equiv \frac{\partial\mathcal{L}}{\partial z^i} = g_{i\bar{j}}\dot{\bar{z}}^{\bar{j}} - \frac{i}{2}\partial_k g_{i\bar{j}}\bar{\chi}^{\bar{j}}\chi^k\\
\bar{P}_{\bar{j}} &\equiv \frac{\partial\mathcal{L}}{\partial\bar{z}^{\bar{j}}} = g_{i\bar{j}}\dot{z}^i + \frac{i}{2}\partial_{\bar{l}}g_{i\bar{j}}\bar{\chi}^{\bar{l}}\chi^i\\
\pi_{i\alpha} &\equiv \frac{\partial\mathcal{L}}{\partial\chi^{i\alpha}} = -\frac{i}{2}g_{i\bar{j}}\bar{\chi}^{\bar{j}}_\alpha\\
\bar{\pi}_{\bar{j}\alpha} &\equiv \frac{\partial\mathcal{L}}{\partial\bar{\chi}^{\bar{j}\alpha}} = +\frac{i}{2}g_{i\bar{j}}\chi^i_\alpha
\end{aligned} \tag{3.19}$$

And their canonical Poisson brackets are

$$\begin{aligned}
\{z^k, P_i\} &= \delta^i_k & \{\bar{z}^{\bar{j}}, \bar{P}_{\bar{l}}\} &= \delta^{\bar{j}}_{\bar{l}}\\
\{\chi^{i\alpha}, \pi_{k\beta}\} &= -\delta^i_k\delta^\alpha_\beta & \{\bar{\chi}^{\bar{j}\alpha}, \bar{\pi}_{\bar{l}\beta}\} &= -\delta^{\bar{j}}_{\bar{l}}\delta^\alpha_\beta\,.
\end{aligned} \tag{3.20}$$

This system possesses fermionic constraints of the second kind

$$\phi^\alpha_i \equiv \pi^\alpha_i + \frac{i}{2}g_{i\bar{j}}\bar{\chi}^{\bar{j}\alpha} = 0 \qquad \bar{\phi}_{\bar{j}\alpha} \equiv \bar{\pi}_{\bar{j}\alpha} - \frac{i}{2}g_{i\bar{j}}\chi^i_\alpha = 0\,. \tag{3.21}$$

JHEP10(2000)024

whose Poisson bracket is

$$\{\phi_i^\alpha, \bar{\phi}_{\bar{j}\beta}\} = -i g_{i\bar{j}} \delta_\beta^\alpha \,. \tag{3.22}$$

Following standard procedure [31], we define the Dirac brackets

$$\{\circ,\circ\}_D \equiv \{\circ,\circ\} - \{\circ,\phi_i^\alpha\}\frac{1}{\{\phi_i^\alpha,\bar{\phi}_{\bar{j}\beta}\}}\{\bar{\phi}_{\bar{j}\beta},\circ\} - \{\circ,\bar{\phi}_{\bar{j}\beta}\}\frac{1}{\{\bar{\phi}_{\bar{j}\beta},\phi_i^\alpha\}}\{\phi_i^\alpha,\circ\}\,. \tag{3.23}$$

Canonical quantization proceeds by substituting $\{\circ,\circ\}_D \to -i[\circ,\circ]$ or $-i\{\circ,\circ\}$, as appropriate. Evaluating the Dirac brackets for all quantities, we obtain the following algebra

$$[z^k, P_i] = i\delta_i^k \,, \qquad [\bar{z}^{\bar{l}}, \bar{P}_{\bar{j}}] = i\delta_{\bar{j}}^{\bar{l}} \,, \qquad \{\chi^{i\alpha}, \bar{\chi}_\beta^{\bar{j}}\} = g^{i\bar{j}}\delta_\beta^\alpha \,. \tag{3.24}$$

The commutators of z_i and $\bar{z}_{\bar{j}}$ with the fermions $\chi^{i\alpha}$ and $\bar{\chi}^{\bar{j}\beta}$ vanish. Define

$$K_i \equiv \frac{i}{2}\partial_k g_{i\bar{j}} \bar{\chi}^{\bar{j}\beta}\chi_\beta^k \qquad \bar{K}_{\bar{j}} \equiv -\frac{i}{2}\partial_{\bar{l}} g_{i\bar{j}} \bar{\chi}^{\bar{l}\beta}\chi_\beta^i \,. \tag{3.25}$$

The commutation relationships of P_k and $\bar{P}_j$ alone are not as relevant to what will follow as the commutation relationships involving $P + K$ and $\bar{P} + \bar{K}$. With some work, it can be shown that

$$\begin{aligned} &[(P+K)_k, \chi^{i\alpha}] = i\Gamma^i_{ka}\chi^{a\alpha} \qquad && [(P+K)_k, \bar{\chi}^{\bar{j}\alpha}] = 0 \\ &[(\bar{P}+\bar{K})_{\bar{l}}, \chi^{i\alpha}] = 0 && [(\bar{P}+\bar{K})_{\bar{l}}, \bar{\chi}^{\bar{j}\alpha}] = i\bar{\Gamma}^{\bar{j}}_{\bar{l}\bar{b}}\bar{\chi}^{\bar{b}\alpha} \end{aligned} \tag{3.26}$$

and that

$$\begin{aligned} &[(P+K)_i, (P+K)_k] = [(\bar{P}+\bar{K})_{\bar{j}}, (\bar{P}+\bar{K})_{\bar{l}}] = 0 \\ &[(P+K)_i, (\bar{P}+\bar{K})_{\bar{j}}] = -R_{i\bar{j}a\bar{b}}\chi^a\bar{\chi}^{\bar{b}} \,. \end{aligned} \tag{3.27}$$

The (classical) hamiltonian obtained by a standard procedure is

$$H = g^{i\bar{j}}(P+K)_i(\bar{P}+\bar{K})_{\bar{j}} + \frac{1}{4}R_{i\bar{j}a\bar{b}}(\chi^i\chi^a)\left(\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{b}}\right) . \tag{3.28}$$

Since $R_{a\bar{b}} = g^{i\bar{j}}R_{i\bar{j}a\bar{b}} = 0$, there are no ordering ambiguities and thus (3.28) can be simply taken to be the quantum hamiltonian. A very important cyclic identity for the fermionic coordinates is that

$$\chi^\alpha(\psi\phi) + \psi^\alpha(\phi\chi) + \phi^\alpha(\chi\psi) = 0 \,. \tag{3.29}$$

This identity must be used with care for the quantum fermionic coordinates, since they have non-trivial anti-commutation relationships. Nevertheless, the hamiltonian can be rewritten as

$$H = g^{i\bar{j}}(p+K)_i(\bar{p}+\bar{K})_{\bar{j}} - \frac{1}{2}R_{i\bar{j}a\bar{b}}\left(\chi^i\bar{\chi}^{\bar{j}}\right)\left(\chi^a\bar{\chi}^{\bar{b}}\right) . \tag{3.30}$$

The (classical) supersymmetry transformations are given in [30]

$$\delta_\xi z^i = \xi\chi^i\,, \qquad \delta_\xi \chi^{i\alpha} = -i\bar{\xi}^\alpha \dot{z}^i - \Gamma^i_{jk}(\xi\chi^j)\chi^{k\alpha}\,. \tag{3.31}$$

With some work, the Nöether current can be calculated for the real form of the lagrangian (3.18) to be

$$J_{\rm SUSY} = g_{i\bar{j}}\dot{\bar{z}}^{\bar{j}}\xi\chi^i - g_{i\bar{j}}\dot{z}^i\bar{\xi}\bar{\chi}^{\bar{j}}\,. \tag{3.32}$$

The minus sign is related to the minus sign in equation (3.17). Defining the SUSY generators through $J_{\rm SUSY} = Q^\alpha\xi_\alpha + \bar{Q}_\alpha\bar{\xi}^\alpha$ we obtain that

$$Q^\alpha = g_{i\bar{j}}\dot{\bar{z}}^{\bar{j}}\chi^{i\alpha} = (P+K)_i\chi^{i\alpha}\,, \qquad \bar{Q}_\alpha = g_{i\bar{j}}\dot{z}^i\bar{\xi}\bar{\chi}^{\bar{j}} = (\bar{P}+\bar{K})_{\bar{j}}\bar{\chi}^{\bar{j}}_\alpha\,. \tag{3.33}$$

There are no ordering ambiguities when interpreting these as quantum operators.

It is now a matter of another careful computation to confirm that the Q's satisfy the SUSY algebra

$$\{Q^\alpha, Q^\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0\,, \qquad \{Q^\alpha, \bar{Q}_\beta\} = \delta^\alpha_\beta H\,. \tag{3.34}$$

We must represent this algebra in a coordinate basis. There are 4 fermionic coordinates and thus the Hilbert space can be written as a $2^4 = 16$-dimensional vector of complex functions of z and $\bar{z}$. Equation (3.24) suggests that we associate P_i with $-i\partial_i$ and $\bar{P}_{\bar{j}}$ with $-i\partial_{\bar{j}}$. Naively, this would disagree with (3.26) and (3.27). We need to pay attention to how these partial derivatives act on the fermionic basis.

Let us begin by noticing that

$$[(P+K)_k, \chi^\alpha_{\bar{j}}] = [(P+K)_k, \bar{\chi}^{\bar{j}\alpha}] = 0\,, \qquad \{\chi^\alpha_{\bar{l}}, \bar{\chi}^{\bar{j}}_\beta\} = \delta^{\bar{j}}_{\bar{l}}\delta^\alpha_\beta \tag{3.35}$$

and

$$[(\bar{P}+\bar{K})_{\bar{l}}, \chi^{i\alpha}] = [(\bar{P}+\bar{K})_{\bar{l}}, \bar{\chi}^\alpha_i] = 0\,, \qquad \{\chi^{i\alpha}, \bar{\chi}_{k\beta}\} = \delta^i_k\delta^\alpha_\beta \tag{3.36}$$

Define $|\downarrow\rangle$ by $\bar{\chi}^{\bar{j}}_\beta|\downarrow\rangle = \bar{\chi}_{i\beta}|\downarrow\rangle = 0$. It is consistent with all commutation relationships to write that $(P+K)_i[f(z,\bar{z})|\downarrow\rangle] = (-i\partial_i f(z,\bar{z}))|\downarrow\rangle$ and $(\bar{P}+\bar{K})_{\bar{j}}[f(z,\bar{z})|\downarrow\rangle] = -i\partial_{\bar{j}}f(z,\bar{z})|\downarrow\rangle$ From the no-fermions state $|\downarrow\rangle$ we can construct the one-fermion state in two ways: using $\chi^{i\alpha}$ or $\chi^\alpha_{\bar{j}}$. Let $|\chi^{i\alpha}\rangle \equiv \chi^{i\alpha}|\downarrow\rangle$ and $|\chi^\alpha_{\bar{j}}\rangle \equiv \chi^\alpha_{\bar{j}}|\downarrow\rangle = g_{i\bar{j}}\chi^{i\alpha}|\downarrow\rangle$. It can be then checked by an explicit calculation that the following is consistent with all commutation relationships

$$\begin{aligned}(P+K)_i[f^{\bar{j}}_\alpha(z,\bar{z})|\chi^\alpha_{\bar{j}}\rangle] &= (-i\partial_i f^{\bar{j}}_\alpha(z,\bar{z}))|\chi^\alpha_{\bar{j}}\rangle\\ (\bar{P}+\bar{K})_{\bar{j}}[f_{i\alpha}(z,\bar{z})|\chi^{i\alpha}\rangle] &= (-i\partial_{\bar{j}}f_{i\alpha}(z,\bar{z}))|\chi^{i\alpha}\rangle\,.\end{aligned} \tag{3.37}$$

Thus, one should think of $|\downarrow\rangle$ as being independent of both z and $\bar{z}$, of $\chi^{i\alpha}$ and $\bar{\chi}^\alpha_i$ as being holomorphic functions and $\chi^\alpha_{\bar{j}}$ and $\bar{\chi}^{\bar{j}\alpha}$ as being antiholomorphic functions.

We can continue this procedure by defining two, three and four-fermion states $|\chi\chi\rangle$, $|\chi\chi\chi\rangle$ and $|\chi\chi\chi\chi\rangle$, and checking each time that the commutation relationships work. The computation is made easier if one notices that the four-fermion state $|\chi\chi\chi\chi\rangle$ is the same as $|\uparrow\rangle$ defined by $\chi^{\beta}_{\bar{j}}|\uparrow\rangle = \chi^{i\beta}|\uparrow\rangle = 0$ and that $|\chi\chi\chi\rangle$ can be written as $|\bar{\chi}\rangle \equiv \bar{\chi}|\uparrow\rangle$. With this explicit construction, we can write the energy eigenvalue equation $H|\rangle = E|\rangle$ as a differential equation. Notice that H commutes with the fermion number operator $\chi^i\bar{\chi}_i$ and so we can write different differential equations for each fermion number. For the no-fermions state, $f|\downarrow\rangle$, $H|\rangle = E|\rangle$ can be written as

$$-g^{i\bar{j}}\partial_i\partial_{\bar{j}} f(z,\bar{z}) = E f(z,\bar{z})\,. \tag{3.38}$$

For the one-fermion states, $f_{i\alpha}(z,\bar{z})|\chi^{i\alpha}\rangle$, the equation is

$$-g^{k\bar{j}}\partial_k\partial_{\bar{j}} f_{i\alpha} + g^{k\bar{j}}\Gamma^a_{ik}\partial_{\bar{j}} f_{a\alpha} = E f_{i\alpha}\,. \tag{3.39}$$

And, finally, for the two-fermion states, $f_{nm\alpha\beta}(z,\bar{z})|\chi^{n\alpha}\chi^{m\beta}\rangle$, we obtain

$$-g^{k\bar{j}}\partial_k\partial_{\bar{j}} f_{nm\alpha\beta} + g^{k\bar{j}}(\Gamma^a_{nk}\partial_{\bar{j}} f_{am\alpha\beta} + \Gamma^a_{mk}\partial_{\bar{j}} f_{na\alpha\beta}) - R_{n\bar{b}m\bar{d}}\, g^{a\bar{b}} g^{c\bar{d}} f_{ac\alpha\beta} = E f_{nm\alpha\beta}\,. \tag{3.40}$$

The equations for three- and four-fermion states can be obtained by analogy to (3.39) and (3.38).

Given the tensor J defined in equation (3.14), we obtain four more supersymmetry generators

$$S^{\alpha} \equiv (\bar{P}+\bar{K})_{\bar{j}} J^{\bar{j}}_{i}\chi^{i\alpha} \qquad\qquad \bar{S}_{\alpha} \equiv (P+K)_i J^{i}_{\bar{j}}\bar{\chi}^{\bar{j}}_{\alpha}\,. \tag{3.41}$$

These can be confirmed to satisfy $\mathcal{N}=8$ SUSY algebra, namely

$$\{Q^{\alpha},\bar{Q}_{\beta}\} = \{S^{\alpha},S_{\beta}\} = \delta^{\alpha}_{\beta} H \tag{3.42}$$

and

$$\begin{aligned}\{Q^{\alpha},\bar{S}_{\beta}\} = \{S^{\alpha},\bar{Q}_{\beta}\} = \{Q^{\alpha},Q^{\beta}\} = \{S^{\alpha},Q^{\beta}\} = \\ \{S^{\alpha},S^{\beta}\} = \{\bar{Q}_{\alpha},\bar{Q}_{\beta}\} = \{\bar{S}_{\alpha},\bar{Q}_{\beta}\} = \{\bar{S}_{\alpha},\bar{S}_{\beta}\} = 0\end{aligned} \tag{3.43}$$

In proving the above relationships, we need to use the properties of J: the metric is hermitean, J is covariantly constant and the Nijenhuis tensor, $\mathcal{N}(J)$, vanishes.

3.3 Scattering

Consider the simplest scattering question — s-wave scattering of one of the scalar particles in the low energy regime. This is described by $-\nabla^2 f = Ef$. The laplacian was given in equation (3.9). For s-wave scattering, $f = f(R)$ and the differential equation to be solved is

$$-\frac{1}{R^3}\partial_R\left(\frac{R^4-1}{R}\partial_R f(R)\right) = \xi^2 E f(R)\,. \tag{3.44}$$

Let $y^4 = \xi^2 E \ll 1$ and define $x = yR$. We can then rewrite (3.44) as

$$-\frac{1}{x^3}\partial_x\left(\frac{x^4-y^4}{x}\partial_x f(x)\right) = y^2 f(x)\,. \tag{3.45}$$

For $x \ll 1/y$ we can treat the r.h.s. of equation (3.45) as a small perturbation. Keeping the solution finite at x=y, we obtain a solution as an expansion in y

$$1 - \frac{1}{8}x^2y^2 + \frac{1}{192}x^4y^4 - \frac{1}{9216}x^6y^6 + o(y^8)\,. \tag{3.46}$$

For $x \gg y$, rewrite (3.45) as

$$-\frac{1}{x^3}\partial_x\left(x^3\partial_x f(x)\right) - y^2 f(x) = -y^4\frac{1}{x^3}\partial_x\left(\frac{1}{x}\partial_x f(x)\right)\,. \tag{3.47}$$

The r.h.s. is again a small perturbation. The solution with the r.h.s. set to zero is the asymptotic solution at infinity

$$\frac{2}{y}\frac{J_1(yx) + aY_1(yx)}{x} \tag{3.48}$$

(the factor $(2/y)$ is for convenience only). We need to obtain the lowest order correction due to the r.h.s., which is $-y^6/24x^2$. The expansion in y is thus

$$1 - \frac{1}{8}x^2y^2 + \frac{1}{192}x^4y^4 - \left(\frac{1}{9216}x^6 + \frac{1}{24x^2}\right)y^6 + o(y^8) - \frac{4a}{\pi y^2 x^2} + \cdots\,. \tag{3.49}$$

Since the region of validity of the two expansions (3.46) and (3.49) in y overlaps for $x \sim 1$, they should match. We need to cancel the x^{-2} terms in expansion (3.49), and thus

$$a = -\frac{\pi y^8}{96} = -\frac{\pi\xi^4E^2}{96}\,. \tag{3.50}$$

3.4 The SO(5) symmetry and two-particle states

The tensor multiplet on the M5-brane contains 5 real scalars with an SO(5) symmetry. This symmetry should somehow be visible in the M(atrix)-model we have just constructed. In this section we will find the SO(5) symmetry which commutes with the hamiltonian and construct a real five-dimentional representation of it.

The SO(5) symmetry we are looking for contains the $\mathrm{SO}(3)_\Omega$ symmetry acting naturally on the three Kähler forms Ω_i's, as well as the $\mathrm{SU}(2) = \mathrm{SO}(3)_f$ symmetry acting on the fermionic indices. Since the SO(5) symmetry is connected to the scalars, it should act non-trivially on the states $f|\downarrow\rangle$, $f|\uparrow\rangle$ and $|\chi^{n\alpha}\chi^{m\beta}\rangle$. Let us see what is the action of the $\mathrm{SO}(3)_\Omega$ symmetry on the fermion vacuum states. For example, consider using $\Omega_1 + \epsilon\Omega_2$ instead of Ω_1 to define the complex coordinates $(z^i, \bar{z}^{\bar{j}})$. The required change of coordinates is

$$dz^i \to dz^i + \epsilon g^{i\bar{j}}(\Omega_2)_{\bar{j}\bar{k}}d\bar{z}^{\bar{k}}\,, \qquad d\bar{z}^{\bar{i}} \to d\bar{z}^{\bar{i}} + \epsilon g^{\bar{i}j}(\Omega_2)_{jk}dz^k\,. \tag{3.51}$$

This being just a change of variables, we know how it will act on the fermionic variables $\chi^{i\alpha}$

$$\chi^{i\alpha} \to \chi^{i\alpha} + \epsilon g^{i\bar{j}}(\Omega_2)_{\bar{j}\bar{k}}\bar{\chi}^{\bar{k}}_{\alpha}\,, \qquad \bar{\chi}^{\bar{i}}_{\alpha} \to \bar{\chi}^{\bar{i}}_{\alpha} + \epsilon g^{\bar{i}j}(\Omega_2)_{jk}\chi^{k\alpha}\,. \tag{3.52}$$

The new $\bar{\chi}$'s must annihilate the new vacuum, and so we infer that

$$|\downarrow\rangle \to |\downarrow\rangle - \frac{1}{2}\epsilon(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle\,, \tag{3.53}$$

where $1, 2$ are concrete fermionic indices. Similarly, we obtain the action on $|\uparrow\rangle$

$$|\uparrow\rangle \to |\uparrow\rangle - \frac{1}{2}\epsilon(\Omega_2)_{\bar{i}\bar{j}}\left(\bar{\chi}^{\bar{i}1}\bar{\chi}^{\bar{j}1} + \bar{\chi}^{\bar{i}2}\bar{\chi}^{\bar{j}2}\right)|\uparrow\rangle\,, \tag{3.54}$$

which can be rewritten as

$$|\uparrow\rangle \to |\uparrow\rangle - \frac{1}{2}\epsilon(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle \tag{3.55}$$

if we define that

$$|\uparrow\rangle \equiv \chi^{11}\chi^{21}\chi^{12}\chi^{22}|\downarrow\rangle\,. \tag{3.56}$$

We also need

$$(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle \to (\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle + 4\epsilon\left(|\uparrow\rangle + |\downarrow\rangle\right). \tag{3.57}$$

Repeating this calculation for the change of coordinates generated by using $\Omega_1 + \epsilon\Omega_3$ instead of Ω_1,we obtain that the following three states form a basis for a real three dimentional representation of $\mathrm{SO}(3)_\Omega$

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)\\
&\frac{1}{2\sqrt{2}}(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle\\
&\frac{i}{\sqrt{2}}\left(|\downarrow\rangle - |\uparrow\rangle\right)
\end{aligned} \tag{3.58}$$

The first and third states are singlets under $\mathrm{SO}(3)_f$, but the second one is not - it is part of a triplet. Filling in the triplet gives us the complete basis for the SO(5) representation we were after

$$\begin{aligned}
|\phi^0\rangle &\equiv \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)\\
|\phi^1\rangle &\equiv \frac{i}{2\sqrt{2}}(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} - \chi^{i2}\chi^{j2}\right)|\downarrow\rangle\\
|\phi^2\rangle &\equiv \frac{1}{2\sqrt{2}}(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle\\
|\phi^3\rangle &\equiv \frac{-i}{2\sqrt{2}}(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j2} + \chi^{i2}\chi^{j1}\right)|\downarrow\rangle\\
|\phi^4\rangle &\equiv \frac{i}{\sqrt{2}}\left(|\downarrow\rangle - |\uparrow\rangle\right).
\end{aligned} \tag{3.59}$$

By construction the SO(5) transformations must commute with the hamiltonian. This manifests itself in the fact that

$$H\left(f_i(z,\bar{z})|\phi^i\rangle\right) = \left(-\nabla^2 f_i(z,\bar{z})\right)|\phi^i\rangle \tag{3.60}$$

as can be checked explicitly. All five of the scalars have the same equation of motion. We now know which of the 6 two-fermion states $|\chi\chi\rangle$ correspond to scalars and which to the 3-form field. The later is represented by the singlet states under $SO(3)_f$, namely by $f_{ij}\epsilon_{\alpha\beta}|\chi^{i\alpha}\chi^{j\beta}\rangle$ where $f_{ij} = f_{ji}$ has three independent components as needed.

3.5 Relation to the M(atrix)-model

The calculation that we performed should be compared with the $O(\Theta)^2$ scattering in the M(atrix)-model of the noncommutative tensor multiplet. We will not perform the comparison in detail here [32]. For the scattering of two massless particles, there are 4 Feynman diagrams that contribute at $O(\Theta)^2$. One from the quartic vertex (2.4) and the other contributions are from the tree diagrams (the s, t, u channels) with two cubic $\Phi\Phi H$ vertices as in (2.3) and an $[HH]$ propagator.

Let us see what does this all mean for scattering of two scalar particles on the M5-brane. Let the fermionic coordinates corresponding to the centre-of-mass coordinates be denoted by $\psi^{i\alpha}$, and those corresponding to the individual particles be $A^{i\alpha} = (\chi^{i\alpha} + \psi^{i\alpha})/2$ and $B^{i\alpha} = (\chi^{i\alpha} - \psi^{i\alpha})/2$. Since for scattering we are only interested in asymptotic states, we can assume that everything happens on a flat manifold. Now, a state of two identical scalars A and B is just $|\phi^0\rangle_A|\phi^0\rangle_B$ where we chose a specific direction under the SO(5). This can be rewritten in terms of the separated coordinates χ and ϕ

$$\begin{aligned} |\phi^0\rangle_A|\phi^0\rangle_B &= \frac{1}{2}\left[|\downarrow\rangle_A|\downarrow\rangle_B + |\uparrow\rangle_A|\uparrow\rangle_B + (|\downarrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B)\right] \\ &= \frac{1}{2}\left(|\downarrow\rangle_\phi|\downarrow\rangle_\chi + |\uparrow\rangle_\phi|\uparrow\rangle_\chi\right) + \sum |\rangle_\phi|\text{scalar}\rangle_\chi + \sum |\rangle_\phi|\text{3-form}\rangle_\chi\,. \end{aligned} \tag{3.61}$$

Choosing another-two particle state, we obtain

$$\begin{aligned} |\phi^4\rangle_A|\phi^4\rangle_B &= -\frac{1}{2}\left[|\downarrow\rangle_A|\downarrow\rangle_B + |\uparrow\rangle_A|\uparrow\rangle_B - (|\downarrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B)\right] \\ &= -\frac{1}{2}\left(|\downarrow\rangle_\phi|\downarrow\rangle_\chi + |\uparrow\rangle_\phi|\uparrow\rangle_\chi\right) + \sum |\rangle_\phi|\text{scalar}\rangle_\chi + \sum |\rangle_\phi|\text{3-form}\rangle_\chi\,. \end{aligned} \tag{3.62}$$

Thus, the scattering matrix for $\psi^I\psi^I \to \psi^J\psi^J$ will have a form

$$_A\langle\phi^4| \quad _B\langle\phi^4| \quad S \quad |\phi^0\rangle_A \quad |\phi^0\rangle_B = -\frac{1}{2}\sigma_1 + \sigma_2 + \sigma_3 \tag{3.63}$$

and that for $\psi^I\psi^I \to \psi^I\psi^I$ will be

$$_A\langle\phi^4| \quad _B\langle\phi^4| \quad S \quad |\phi^0\rangle_A \quad |\phi^0\rangle_B = \frac{1}{2}\sigma_1 + \sigma_2 + \sigma_3\,, \tag{3.64}$$

where

$$
\begin{aligned}
\sigma_1 &= {}_\chi\langle\downarrow|S|\downarrow\rangle_\chi = {}_\chi\langle\uparrow|S|\uparrow\rangle_\chi \\
\sigma_2 &= \left(\sum{}_\chi\langle\text{scalar}|\right) S \left(\sum|\text{scalar}\rangle_\chi\right) \\
\sigma_3 &= \left(\sum{}_\chi\langle\text{3-form}|\right) S \left(\sum|\text{3-form}\rangle_\chi\right)
\end{aligned}
\tag{3.65}
$$

where we have ignored the (trivial) evolution of the centre-of-mass coordinates.

The answers are as we would expect: the difference (equal to σ_1) between the matrix element for $\psi^I\psi^I \to \psi^I\psi^I$ and for $\psi^I\psi^I \to \psi^J\psi^J$ arises to the lowest order in the effective theory from the extra four-scalar vertex $\psi^I\psi^I\psi^J\psi^J$ which is zero for $I = J$. The other two pieces, σ_2 and σ_3 will have different momentum behaviour, as can be seen in an explicit Feynmann diagram computation.

It is interesting to see what happens in the approximation of a large impact parameter. From the M(atrix)-model we expect a force that behaves as v^2/r^4 where r is the distance between the particles and v is the relative transverse velocity. In the large impact parameter approximation, the t-channel dominates. From (2.2) we see that the H-propagator behaves as $1/r^4$ when there is no longitudinal momentum transfer. The two $\Phi\Phi H$ vertices should contribute a v^2.

Acknowledgments

We are very grateful to M. Berkooz for discussions. We also wish to thank O. Aharony and A. Volovich for comments on a previous version. The research of OJG is supported by NSF grant number PHY-9802498. The research of JLK is in part supported by NSERC (Natural Sciences and Engineering Research Council of Canada).

A. Scattering in the field theory

In this appendix we will describe how to calculate the scattering in field theory. Let us consider the scattering of two scalars to lowest nontrivial order. There are two contributions. One from the $O(\Theta)^2$ quartic vertex and one from a tree diagram with an H-field exchanged. Let us consider the amplitude

$$
A^{I_1 I_2 I_3 I_4}(p_1, p_2, p_3, p_4)\,,
$$

with $p_1 + p_2 = p_3 + p_4$.

The amplitude is a sum of s, t, u channels and a quartic vertex:

$$
A = A_t + A_u + A_s + A_q\,.
$$

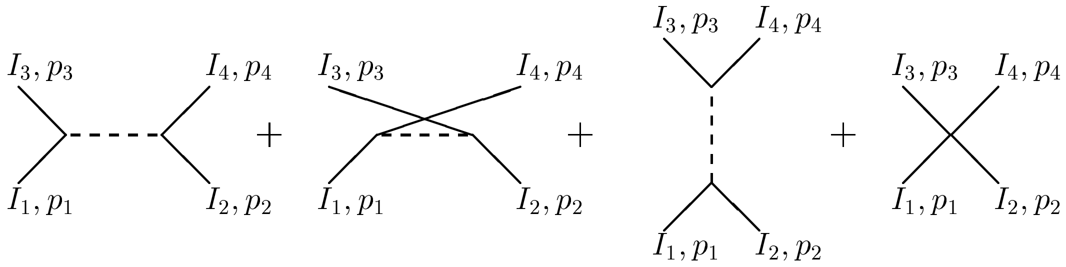
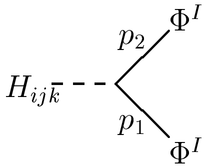


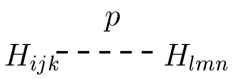
Figure 1: Feynman diagrams contributing to the scattering of two scalars to lowest non-trivial order in θ.

A.1 The Feynman rules

The Feynman rules are as follows.



$$= \Theta^{l[ij}(p_2)^{k]}(p_1)_l + \Theta^{l[ij}(p_1)^{k]}(p_2)_l$$



$$\begin{aligned} &= \frac{36\pi}{p^2+i\epsilon}\eta_{rl}\eta_{sm}\eta_{tn}\left(p_{[i}p^{[r}\delta_j^s\delta_{k]}^{t]} - \frac{1}{6}p_u\epsilon_{ijk}{}^{u[st}p^{r]}\right) + \\ &\quad + \frac{\pi}{2}\left(\epsilon_{ijklmn} - 6\delta_{[i}^{[r}\delta_j^s\delta_{k]}^{t]}\eta_{lr}\eta_{ms}\eta_{nt}\right). \end{aligned}$$

The propagator is:

$$\begin{aligned} \langle H_{ijk}(p)H_{lmn}(-p')\rangle &= \frac{36\pi}{p^2+i\epsilon}\eta_{rl}\eta_{sm}\eta_{tn}\left(p_{[i}p^{[r}\delta_j^s\delta_{k]}^{t]} - \frac{1}{6}p_u\epsilon_{ijk}{}^{u[st}p^{r]}\right)\delta^{(6)}(p-p') + \\ &\quad + \frac{\pi}{2}\left(\epsilon_{ijklmn} - 6\delta_{[i}^{[r}\delta_j^s\delta_{k]}^{t]}\eta_{lr}\eta_{ms}\eta_{nt}\right)\delta^{(6)}(p-p'). \end{aligned}$$

The cubic vertex is:

$$\Lambda_{I_1I_2}^{ijk}(p_1,p_2) = \frac{1}{96\pi}\delta_{I_1I_2}\Theta^{l[ij}(p_2)^{k]}(p_1)_l + \frac{1}{96\pi}\delta_{I_1I_2}\Theta^{l[ij}(p_1)^{k]}(p_2)_l$$

In terms of $p \equiv p_1 + p_2$ and $q \equiv p_1 - p_2$, we can write this as:

$$\Lambda_{I_1I_2}^{ijk}(p,q) = \frac{1}{192\pi}\delta_{I_1I_2}\Theta^{l[ij}p^{k]}p_l + \frac{1}{192\pi}\delta_{I_1I_2}\Theta^{l[ij}q^{k]}q_l$$

The quartic interaction is:

$$\begin{aligned} A_q &= \eta_{pq}\Theta^{klp}\Theta^{ijq}p_{1k}p_{2i}p_{3l}p_{4j}(\delta_{I_1I_4}\delta_{I_2I_3} - \delta_{I_1I_2}\delta_{I_3I_4}) + \\ &\quad + \eta_{pq}\Theta^{klp}\Theta^{ijq}p_{1k}p_{2i}p_{3j}p_{4l}(\delta_{I_1I_3}\delta_{I_2I_4} - \delta_{I_1I_2}\delta_{I_3I_4}) + \\ &\quad + \eta_{pq}\Theta^{klp}\Theta^{ijq}p_{1k}p_{2l}p_{3i}p_{4j}(\delta_{I_1I_4}\delta_{I_2I_3} - \delta_{I_1I_3}\delta_{I_2I_4}) + \end{aligned}$$

Let us define:

$$p \equiv p_1 + p_2\,, \qquad q \equiv p_1 - p_2\,, \qquad r \equiv p_3 - p_4\,.$$

We have:

$$p^2 = -q^2 = -r^2\,, \qquad p\cdot q = p\cdot r = 0\,.$$

We get $A_s = A'_s \delta_{I_1 I_2}\delta_{I_3 I_4}$ with

$$\begin{aligned} A'_s = &+\frac{\pi}{2}\eta_{cf}\Theta^{abc}\Theta^{def}p_a r_b p_d r_e + \frac{\pi}{4}\eta_{cf}\Theta^{abc}\Theta^{def}p_a q_b p_d q_e + \\ &+\frac{\pi}{4}p^2\eta_{be}\eta_{cf}\Theta^{abc}\Theta^{def}p_a p_d - \frac{\pi}{16}r_l q^l \eta_{be}\eta_{cf}\Theta^{abc}\Theta^{def}q_a r_d + \\ &+\frac{\pi}{2p^2}\Theta^{abc}\Theta^{def}p_a q_b r_c p_d q_e r_f + \frac{\pi}{2p^2}q_l r^l \eta_{cf}\Theta^{abc}\Theta^{def}p_a q_b p_d r_e\,. \end{aligned} \tag{A.1}$$

The t-channel is given by the same expression (A.1) with

$$p = p_3 - p_1\,, \qquad q = p_3 + p_1\,, \qquad r = p_4 + p_2\,,$$

and the u-channel is given by (A.1) with:

$$p = p_4 - p_1\,, \qquad q = p_4 + p_1\,, \qquad r = p_3 + p_2\,.$$

A.2 Large impact parameters

The approximation of large impact parameter corresponds to $t \gg s, u$. In this case the t-channel amplitude dominates. We can Fourier transform to obtain a force that behaves, at large distances, like v^2/r^4. The H-propagator indeed generates a force that behaves as $1/r^4$ (because it is a harmonic function only in the transverse directions).

In this approximation with keep only the t-channel and we set:

$$\begin{aligned} p_1 &= \left(\vec{v} - \frac{1}{2}\vec{p},\ \frac{R_{||}}{2}\left(\vec{v} - \frac{1}{2}\vec{p}\right)^2,\ \frac{1}{R_{||}}\right) \\ p_2 &= \left(-\vec{v} + \frac{1}{2}\vec{p},\ \frac{R_{||}}{2}\left(\vec{v} - \frac{1}{2}\vec{p}\right)^2,\ \frac{1}{R_{||}}\right) \\ p_3 &= \left(\vec{v} + \frac{1}{2}\vec{p},\ \frac{R_{||}}{2}\left(\vec{v} + \frac{1}{2}\vec{p}\right)^2,\ \frac{1}{R_{||}}\right) \\ p_4 &= \left(-\vec{v} - \frac{1}{2}\vec{p},\ \frac{R_{||}}{2}\left(\vec{v} + \frac{1}{2}\vec{p}\right)^2,\ \frac{1}{R_{||}}\right). \end{aligned}$$

The notation is:

$$p_i = (\vec{p}_i, p_{i-}, p_{i+})\,, \qquad p_i^2 = \vec{p}_i^{\,2} - 2p_{i-}p_{i+}\,.$$

and $R_{||}$ is the radius of the light-like direction of M(atrix)-theory. We set:

$$\begin{aligned} p &= (\vec{p},\ R_{||}(\vec{v}\cdot\vec{p}),\ 0) \\ q &= \left(2\vec{v},\ R_{||}\left(\vec{v}^{\,2} + \frac{1}{4}\vec{p}^{\,2}\right),\ \frac{2}{R_{||}}\right) \\ r &= \left(-2\vec{v},\ R_{||}\left(\vec{v}^{\,2} + \frac{1}{4}\vec{p}^{\,2}\right),\ \frac{2}{R_{||}}\right), \end{aligned}$$

and assume that $|\vec{p}| \ll |\vec{v}|$.

Following [1, 2, 3], we set the nonzero components of Θ to be $\Theta^{ab+} \equiv \theta^{ab}$, with $a, b = 1, \ldots, 4$ and θ is anti-self-dual on $\mathbb{R}^4$. We find that the amplitude is proportional to:

$$A \propto \frac{32\pi}{R_{||}^2 p^2}\theta^{ab} p_a v_b \theta^{de} p_d v_e + \frac{6\pi}{R_{||}^2 p^2}\vec{v}^2 \eta_{cf}\theta^{ac}\theta^{df} p_a p_d. \tag{A.2}$$

After a Fourier transform with respect to $\vec{p}$, this indeed produces a force that is proportional to v^2/r^4.

References

[1] O. Aharony, M. Berkooz and N. Seiberg, *Light-cone description of* (2,0)-*superconformal theories in six dimensions*, *Adv. Theor. Math. Phys.* **2** (1998) 119 [hep-th/9712117].

[2] N. Nekrasov and A. Schwarz, *Instantons on noncommutative* R^4 *and* (2,0)-*superconformal six dimensional theory*, *Comm. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[3] M. Berkooz, *Non-local field theories and the non-commutative torus*, *Phys. Lett.* **B 430** (1998) 237 [hep-th/9802069].

[4] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003, hep-th/9711162.

[5] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[6] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[7] R. Gopakumar, S. Minwalla, N. Seiberg and A. Strominger, *OM theory in diverse dimensions*, *J. High Energy Phys.* **08** (2000) 008 [hep-th/0006062].

[8] H. Verlinde, *A matrix string interpretation of the large-N loop equation*, hep-th/9705029.

[9] S. Gukov, I.R. Klebanov and A.M. Polyakov, *Dynamics of* $(N,1)$ *strings*, *Phys. Lett.* **B 423** (1998) 64 [hep-th/9711112].

[10] N. Seiberg, L. Susskind and N. Toumbas, *Space/time non-commutativity and causality*, *J. High Energy Phys.* **06** (2000) 044 [hep-th/0005015].

[11] E. Bergshoeff, D.S. Berman, J.P. van der Schaar and P. Sundell, *A noncommutative M-theory five-brane*, *Nucl. Phys.* **B 590** (2000) 173 [hep-th/0005026].

JHEP10(2000)024

[12] N. Seiberg, L. Susskind and N. Toumbas, *Strings in background electric field, space/time noncommutativity and a new noncritical string theory*, *J. High Energy Phys.* **06** (2000) 021 [hep-th/0005040].

[13] O.J. Ganor, G. Rajesh and S. Sethi, *Duality and non-commutative gauge theory*, hep-th/0005046.

[14] R. Gopakumar, J. Maldacena, S. Minwalla and A. Strominger, *S-duality and noncommutative gauge theory*, *J. High Energy Phys.* **06** (2000) 036 [hep-th/0005048].

[15] J.L.F. Barbón and E. Rabinovici, *Stringy fuzziness as the custodian of time-space noncommutativity*, *Phys. Lett.* **B 486** (2000) 202 [hep-th/0005073].

[16] J. Gomis and T. Mehen, *Space-time noncommutative field theories and unitarity*, *Nucl. Phys.* **B 591** (2000) 265 [hep-th/0005129].

[17] G.-H. Chen and Y.-S. Wu, *Comments on noncommutative open string theory: V-duality and holography*, hep-th/0006013.

[18] T. Harmark, *Supergravity and space-time non-commutative open string theory*, *J. High Energy Phys.* **07** (2000) 043 [hep-th/0006023].

[19] I.R. Klebanov and J. Maldacena, $1+1$ *dimensional ncos and its* $\mathrm{U}(N)$ *gauge theory dual*, hep-th/0006085.

[20] J.G. Russo and M.M. Sheikh-Jabbari, *On noncommutative open string theories*, *J. High Energy Phys.* **07** (2000) 052 [hep-th/0006202].

[21] O. Aharony, J. Gomis and T. Mehen, *On theories with light-like noncommutativity*, *J. High Energy Phys.* **09** (2000) 023 [hep-th/0006236].

[22] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[23] M.R. Douglas, D. Kabat, P. Pouliot and S.H. Shenker, *D-branes and short distances in string theory*, *Nucl. Phys.* **B 485** (1997) 85 [hep-th/9608024].

[24] S. Paban, S. Sethi and M. Stern, *Constraints from extended supersymmetry in quantum mechanics*, *Nucl. Phys.* **B 534** (1998) 137 [hep-th/9805018].

[25] W. Taylor and M. van Raamsdonk, *Multiple D0-branes in weakly curved backgrounds*, *Nucl. Phys.* **B 558** (1999) 63 [hep-th/9904095].

[26] O.J. Ganor and L. Motl, *Equations of the* $(2,0)$*-theory and knitted five-branes*, *J. High Energy Phys.* **05** (1998) 009, [hep-th/9803108].

[27] M. Kreuzer, J.-G. Zhou, Λ*-symmetry and background independence of noncommutative gauge theory on* R^n, *J. High Energy Phys.* **01** (2000) 011 [hep-th/9912174].

[28] C.M. Hull, A. Karlhede, U. Lindström and M. Rocek, *Nonlinear σ-models and their gauging in and out of superspace*, *Nucl. Phys.* **B 266** (1986) 1.

[29] E.E. Donets, A. Pashnev, J.J. Rosales and M.M. Tsulaia, $N = 4$ *supersymmetric multidimensional quantum mechanics, partial SUSY breaking and superconformal quantum mechanics*, *Phys. Rev.* **D 61** (2000) 043512 [hep-th/9907224].

[30] J.Wess and J.Bagger, *Supersymmetry and supergravity*, Princeton University Press, 1991.

[31] R. Casalbuoni, *The classical mechanics for Bose-Fermi systems*, *Nuovo Cim.* **A33** (1976) 389.

[32] Work in progress.